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Dressing the giant magnon

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Abstract: We apply the dressing method to construct new classical string solutions describing various scattering and bound states of magnons. These solutions carry one, two or three SO(6) charges and correspond to multi-soliton configurations in the generalized sine-Gordon models.

Keywords: Long strings, Integrable Equations in Physics, AdS-CFT Correspondence.

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Contents

1. Introduction

Integrability on both sides of AdS/CFT [1–4] has enabled many quantitative checks of the correspondence (see [5–9] for reviews). At weak 't Hooft coupling, anomalous dimensions of gauge theory operators can be calculated using the Bethe ansatz of an integrable spin chain. At strong coupling, string theory becomes tractable in the semiclassical limit where one can study the energies of the corresponding classical string configurations. Understanding in detail the interpolation between weak and strong coupling remains an outstanding problem.

Recently Hofman and Maldacena [10] suggested a particular limit where the spectrum simplifies on both sides of the correspondence. They considered operators with infinite energy Δ and U(1) R-charge J but finite $\Delta - J$ and fixed 'worldsheet' momentum p. The simplest nontrivial example of such an operator is

$$\mathcal{O}_p \sim \sum_l e^{ipl} \, (\cdots ZZZWZZZ\cdots), \tag{1.1}$$

where Z is a scalar field with R-charge $J = 1$ and W is an excitation inserted at position l along the chain. Note that this is a formal limit where the operator becomes infinitely long and we thus ignore taking the trace (and hence we ignore the cyclicity constraint which would normally set the total momentum to zero).

This limit is different from the BMN limit [11] and has the nice feature that it decouples quantum effects characterized by the 't Hooft coupling λ from finite J effects [12, 13, 10]. In this limit the spectrum on both sides can be analyzed in terms of asymptotic states and the S-matrix describing their scattering. The general state can have any number of elementary magnons as well as bound states thereof.

Hofman and Maldacena identified the elementary magnon (1.1) with a particular string configuration moving on an $\mathbb{R} \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$, which they called the 'giant magnon.' Classical string theory on $\mathbb{R} \times S^2$ is equivalent to classical sine-Gordon theory [14 − 17], and the giant magnon solution of [10] corresponds to the sine-Gordon soliton. Using this map to sine-Gordon theory, the scattering phase of two magnons was computed and shown to match the large λ limit of the conjecture of [18].

In [19] a solution describing a giant magnon moving on $\mathbb{R} \times S^3$ with two angular momenta was constructed, after the existence of such a state had been shown, and a particular case considered, in [20]. The two-charge giant magnon has infinite J just like (1.1), and in addition carries some finite amount J_2 of angular momentum in an orthogonal plane. This solution was obtained by exploiting the correspondence between classical string theory on $\mathbb{R} \times S^3$ and the complex sine-Gordon model. In contrast to (1.1), it corresponds not to a single excitation W but to a bound state of many such excitations carrying a finite macroscopic amount of J_2 charge. More recent work on giant magnons has considered finite J effects [21], some quantum corrections [22], and giant magnon solutions for β-deformed $\mathrm{AdS}_5 \times S^5$ [23].

The aim of this paper is to lay the foundation for a study of more general giant magnon solutions on $\mathbb{R} \times S^5$. We define a giant magnon to be any open string on $\mathbb{R} \times S^{N-1}$ whose endpoints move at the speed of light along the equator of the sphere. One can build a physical closed string solution from two or more giant magnons by attaching the beginning of each giant magnon to the end of another.

Previous studies [10, 19] have employed the correspondence between classical string theory on $\mathbb{R} \times S^2$ (or $\mathbb{R} \times S^3$) and the sine-Gordon (or complex sine-Gordon) model. More generally, string theory on $\mathbb{R} \times S^{N-1}$ is classically equivalent to the so-called SO(N) SSSG (symmetric space sine-Gordon) model [24, 25]. An advantage of using the sine-Gordon formulation of the problem is that explicit formulas are known for arbitrary n-soliton configurations in these theories. The disadvantage of using the sine-Gordon formulation is that the map between the sine-Gordon variables and the string sigma-model variables X_i describing the embedding of the string into $\mathbb{R} \times S^{N-1}$ is nonlinear and difficult or impossible to invert in practice for any but the simplest configurations.

In this paper we instead focus directly on the SO(N) vector model describing strings on $\mathbb{R} \times S^{N-1}$ and the SU(2) principal chiral model describing strings on $\mathbb{R} \times S^3$. Being integrable, there exists a procedure for directly constructing their soliton solutions. We employ the dressing method [26 − 28] to construct classical string solutions corresponding to various scattering and bound states of magnons, as well as scattering states of bound states.[1]

We use the dressing method to rederive the previously known giant magnon solutions (5.5) and (4.15), which both correspond to single sine-Gordon solitons, and further use it to construct several new solutions (4.18), (4.26), (5.10), (5.14), and (5.19) corresponding to scattering or bound states of two solitons carrying one, two or three SO(6) charges. (Solutions equivalent to (5.10) and (5.14) have also been obtained by J. Maldacena and A. Mikhailov from the Bäcklund transformation [29].) Moreover, as we discuss below, the dressing method allows general n-soliton scattering and bound states to be constructed algebraically.

It is an important open problem to determine an overall λ-dependent phase factor in the magnon S-matrix [30, 31], whose zeros and poles must contain information about the spectrum of magnon bound states. We calculate the dispersion relations for all of the solutions constructed in this paper, but we do not address here the calculation of the scattering phase. At the semiclassical level, it can be computed by simply translating the result from the corresponding sine-Gordon picture, as was done in [10] for two magnons on $\mathbb{R} \times S^2$. The calculation of quantum corrections to the scattering phase would require the explicit formulas presented in (4.18), (5.10) or (5.19) below since the correspondence with the sine-Gordon model breaks down at the quantum level.

We begin in section 2 with a brief statement of our notation and the equations to be solved. In section 3 we review the dressing method for the principal chiral model, and explain how to apply it to the SO(N) vector model by a particular embedding. In sections 4 and 5 we apply this method to construct explicit multi-soliton string configurations for $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^{N-1}$ respectively.

2. Giant magnon preliminaries

We use worldsheet coordinates t (identified with physical time) and x, which ranges from $-\infty$ to $+\infty$. In conformal gauge, a giant magnon is a solution of the sigma model equations of motion (we use $z = \frac{1}{2}(x - t)$, $\bar{z} = \frac{1}{2}(x + t)$)

$$\bar{\partial}\partial X_i + (\partial X_j \bar{\partial} X_j) X_i = 0, \qquad X_i X_i = 1, \tag{2.1}$$

subject to the Virasoro constraints

$$\partial X_i \partial X_i = \bar{\partial} X_i \bar{\partial} X_i = 1. \tag{2.2}$$

When useful, we will employ the complex coordinates

$$Z_1 = X_1 + iX_2, \qquad Z_2 = X_3 + iX_4, \qquad Z_3 = X_5 + iX_6. \tag{2.3}$$

[1] The previous sentence highlights a possible terminological confusion in this subject. In this paper we consider only single giant magnons; that is, single open strings, corresponding to a single operator but with a possibly arbitrary number of magnon excitations W. The notion of 'soliton number' is well-defined in the integrable SO(N) vector model, so we will characterize giant magnon solutions according to how many solitons they carry. Each soliton may correspond to one magnon W or to a bound state of many magnons.

JHEP10(2006)012

The boundary conditions for a giant magnon at fixed t are

$$\begin{aligned} Z_1(t, x \to \pm\infty) &= e^{it \pm ip/2 + i\alpha}, \\ Z_i(t, x \to \pm\infty) &= 0, \qquad i = 2, 3, \end{aligned} \tag{2.4}$$

where α is any real constant and p represents the total worldsheet momentum of the magnon. Geometrically, p represents the difference in longitude between the two endpoints of the string on the equator of the S^5. The first condition (2.4) only defines p modulo 2π. Although this is sufficient for giant magnon states corresponding to a single soliton, more general giant magnons corresponding to scattering or bound states of many solitons can carry arbitrary p. A better definition is

$$p = \frac{1}{i} \int_{-\infty}^{+\infty} dx \, \frac{d}{dx} \log Z_1. \tag{2.5}$$

In addition to p, giant magnons can be characterized by the conserved charges

$$\begin{aligned} \Delta - J &= \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{+\infty} dx \, \left(1 - \mathrm{Im}[\bar{Z}_1 \partial_t Z_1]\right), \\ J_i &= \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{+\infty} dx \, \mathrm{Im}[\bar{Z}_i \partial_t Z_i], \qquad i = 2, 3, \end{aligned} \tag{2.6}$$

where λ is the 't Hooft coupling. Note that Δ and J are separately infinite for a giant magnon; only their difference is finite.

3. Review of the dressing method

In this section we briefly review the dressing method of Zakharov and Mikhailov [26, 27] for constructing soliton solutions of classically integrable equations. This is a very general technique, but we restrict our attention to its application to the principal chiral model, since all of the solutions given in this paper may be embedded into it as discussed below.

We consider a unitary $N \times N$ matrix field $g(z, \bar{z})$ subject to the equation of motion

$$\bar{\partial} \left(\partial g \, g^{-1}\right) + \partial \left(\bar{\partial} g \, g^{-1}\right) = 0. \tag{3.1}$$

The dressing method allow us to start with some given solution g of this equation and construct a new solution g' by

$$g \to g' = \chi g \tag{3.2}$$

for some appropriately chosen χ. If χ were just a constant matrix, this would be an uninteresting unitary transformation, so to generate physically distinct solutions we want χ to depend on z and $\bar{z}$.

3.1 Construction

The dressing method construction proceeds by introducing an auxiliary variable λ (called the spectral parameter, not to be confused with the 't Hooft coupling in (2.6)) and considering the system of equations

$$i\bar{\partial}\Psi = \frac{A\Psi}{1+\lambda}, \qquad i\partial\Psi = \frac{B\Psi}{1-\lambda} \tag{3.3}$$

for three matrices $\Psi(\lambda)$, A, and B (it is crucial that A and B are independent of λ).

The relation between (3.3) and (3.1) is as follows. If we have any solution g to (3.1), then we can take

$$A = i\bar{\partial} g\, g^{-1}, \qquad B = i\partial g\, g^{-1} \tag{3.4}$$

and then solve (3.3) to find $\Psi(\lambda)$ such that

$$\Psi(0) = g. \tag{3.5}$$

On the other hand, suppose we have any collection $(\Psi(\lambda), A, B)$ which satisfies (3.3) for all λ. Then it is easy to check that $\Psi(0)$ is guaranteed to satisfy (3.1). We impose on $\Psi(\lambda)$ the unitarity condition

$$\Psi^\dagger(\bar{\lambda})\Psi(\lambda) = 1. \tag{3.6}$$

Suppose we consider the analogue of the gauge transformation (3.2) for the auxiliary system (3.3), now with a λ-dependent gauge parameter $\chi(\lambda)$,

$$\begin{aligned} \Psi &\to \Psi' = \chi\Psi, \\ A &\to A' = \chi A \chi^{-1} + i(1+\lambda)\bar{\partial}\chi\chi^{-1}, \\ B &\to B' = \chi B \chi^{-1} + i(1-\lambda)\partial\chi\chi^{-1}. \end{aligned} \tag{3.7}$$

If we can arrange for $\chi(\lambda)$ to be chosen in such a way that the new A' and B' remain independent of λ, then $(\Psi'(\lambda), A', B')$ is a legitimate new solution of (3.3), and hence provides a new solution $g' = \Psi'(0)$ of the principal chiral model.

The constraint that A' and B' should be independent of λ is easy to solve by imposing constraints on the analytic properties of $\chi(\lambda)$ in the complex λ-plane. Specifically, we require that $\chi(\lambda)$ should be meromorphic, and that $\chi(\lambda) \to 1$ as $\lambda \to \infty$. We say that $\chi(\lambda)$ has a pole at some λ if any entry of the matrix $\chi(\lambda)$ has a pole there.

Let us demonstrate by means of a simple example how these analyticity constraints may be used to construct the desired $\chi(\lambda)$. In the simplest nontrivial case, $\chi(\lambda)$ has a single pole at some location λ_1. In order to preserve the unitarity condition (3.6), $\chi(\lambda)$ should satisfy

$$\chi^\dagger(\bar{\lambda})\chi(\lambda) = 1. \tag{3.8}$$

Consequently $\chi^{-1}(\lambda)$ must have a single pole at $\bar{\lambda}_1$. Already this information is enough to fix the dressing function χ to be of the form

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P \tag{3.9}$$

where P is a hermitian projection operator (i.e., $P^2 = P = P^\dagger$).

It remains to choose P so that A' and B' are independent of λ. In fact, since they become independent of λ as $\lambda \to \infty$, it is sufficient to check that they have no poles. Looking at (3.7) we might worry that they develop poles at λ_1 (from $\chi(\lambda)$) or $\bar{\lambda}_1$ (from $\chi^{-1}(\lambda)$). It is however easy to check, using the fact that $\Psi(\lambda)$ satisfies the differential equations (3.3), that the residues at these putative poles actually vanish if one chooses the projection operator P such that its image is spanned by a collection of vectors of

JHEP10(2006)012

the form $\{\Psi(\bar{\lambda}_1)e_1, \Psi(\bar{\lambda}_1)e_2, \ldots\}$ where e_i are an arbitrary collection of constant vectors (independent of z and $\bar{z}$). In general the projector P can have any rank, but in all of our applications below P will have rank one, so we write it explicitly as

$$P = \frac{\Psi(\bar{\lambda}_1)ee^\dagger\Psi^{-1}(\lambda_1)}{e^\dagger\Psi^{-1}(\lambda_1)\Psi(\bar{\lambda}_1)e} \tag{3.10}$$

for an arbitrary constant vector e. It is clear that the overall scale of e drops out of (3.10), so in fact e lives in $\mathbb{P}^{N-1}$, which parametrizes the set of lines in $\mathbb{C}^N$. More generally, the data $\{e_i\}$ for a rank k projection operator would be specified by giving an element of the Grassmannian $\mathrm{Gr}(k, N)$ of k-planes in $\mathbb{C}^N$.

3.2 Summary for the U(N) principal chiral model

To summarize, the dressing method proceeds as follows. Given any solution g to the original equation (3.1), we first solve the linear system (3.3) with A and B given by (3.4) to find $\Psi(\lambda)$. The dressed solution $\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda)$ may be constructed using (3.9) and (3.10). Finally, $g' = \Psi'(0)$ provides a new solution of (3.1).

It is clear that successive applications of this simple procedure, i.e. $\Psi''(\lambda) = \chi'(\lambda)\Psi'(\lambda)$ etc., can be used to generate multi-soliton solutions. We will illustrate this construction below via several examples.

3.3 Reduction to the SO(N) vector model

Although the principal chiral model enjoys the most straightforward application of the dressing method, the equations (2.1) describing conformal gauge strings on $\mathbb{R} \times S^{N-1}$ are those of the SO(N) vector model. Imposing the Virasoro constraints (2.2) gives the so-called reduced [14] vector model. We can employ the dressing method for this model by embedding it into the principal chiral model.

We choose the embedding following [32, 28, 33] (a different choice is shown in [34]). Define the diagonal $N \times N$ matrix

$$\theta = \mathrm{diag}(+1, -1, \ldots, -1). \tag{3.11}$$

Then we choose the embedding of the vector X_i into an SO(N) principal chiral field according to the formula

$$\{X_i : X_i X_i = 1\} \qquad \leftrightarrow \qquad g = \theta(2XX^{\mathrm{T}} - 1) \in \mathrm{SO}(N). \tag{3.12}$$

Note that g satisfies the identity

$$g\theta g\theta = 1. \tag{3.13}$$

Geometrically, this identity specifies a particular coset $S^{N-1} = \mathrm{SO}(N)/\mathrm{SO}(N-1)$ sitting inside SO(N). The dressing method proceeds as in the previous subsection, except that we should add to (3.6) the additional conditions [28]

$$\overline{\Psi(\bar{\lambda})} = \Psi(\lambda), \qquad \Psi(\lambda) = \Psi(0)\theta\Psi(1/\lambda)\theta. \tag{3.14}$$

JHEP10(2006)012

In order to preserve (3.14) under dressing, the dressing factor $\chi(\lambda)$ must satisfy

$$\overline{\chi(\bar{\lambda})} = \chi(\lambda), \qquad \chi(\lambda) = \chi(0)\Psi(0)\theta\chi(1/\lambda)\Psi(0)\theta. \tag{3.15}$$

It is not possible for these constraints to be satisfied if $\chi(\lambda)$ has a single pole.

Instead, there are two distinct classes of 'minimal' solitons: the simplest has two poles

$$\text{class I}: \qquad \lambda_1, \quad \bar{\lambda}_1 = 1/\lambda_1 \tag{3.16}$$

located at conjugate points on the unit circle, while the second has four poles located at an arbitrary point λ_1 in the complex plane and its three images under conjugation and inversion,

$$\text{class II}: \qquad \lambda_1, \quad \bar{\lambda}_1, \quad 1/\lambda_1, \quad 1/\bar{\lambda}_1. \tag{3.17}$$

Below we will consider examples of both classes of solitons. We will also consider the case of two class I solitons, with two pairs of conjugate poles on the unit circle, being distinct from a single class II soliton.

For class I the dressing factor is [32, 28]

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1} \overline{P} \tag{3.18}$$

with the projector P given by the same formula (3.10). The constraints (3.15) imply that the constant vector $e \in \mathbb{C}^N$ must satisfy

$$e^{\mathrm{T}} e = 0, \qquad \bar{e} = \theta e. \tag{3.19}$$

The construction of the dressing factor for class II solitons is somewhat more complicated. The reader can find all of the details in Theorem 4.2 and section 5 of [28]. In the example we look at below we will see that the class II soliton with four poles (3.17) in the complex plane can be obtained from an analytic continuation of two pairs of poles on the unit circle describing two class I solitons (3.16).

4. Giant magnons on $\mathbb{R} \times S^3$ from the U(2) principal chiral model

String theory on $\mathbb{R} \times S^3$ admits a particularly simple application of the dressing method since the string equations of motion, in conformal gauge, are equivalent to those of the SU(2) principal chiral model, via the embedding

$$\{(Z_1, Z_2) : |Z_1|^2 + |Z_2|^2 = 1\} \qquad \leftrightarrow \qquad g = \begin{pmatrix} Z_1 & -iZ_2 \\ -i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in \mathrm{SU}(2). \tag{4.1}$$

One minor subtlety which arises for SU groups is that the dressing factor (3.9) does not have unit determinant. Rather,

$$\det \chi(\lambda) = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1}. \tag{4.2}$$

We can ensure that a dressed solution $\chi(0)\Psi(0)$ still sits in SU(2) (rather than U(2)) by throwing in a compensating phase factor $(\bar{\lambda}_1/\lambda_1)^{-1/2}$.

JHEP10(2006)012

4.1 The vacuum

We begin with the vacuum solution

$$\begin{aligned} Z_1 &= e^{it}, \\ Z_2 &= 0 \end{aligned} \tag{4.3}$$

which describes a point-like string moving at the speed of light around the equator of the S^3. This state clearly has $\Delta - J = 0$. Using the embedding (4.1) and (3.4) we find

$$g_0 = \begin{pmatrix} e^{-i(z-\bar{z})} & 0 \\ 0 & e^{+i(z-\bar{z})} \end{pmatrix}, \qquad A_0 = -B_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

The corresponding vacuum solution $\Psi_0(\lambda)$ to the auxiliary problem (3.3) satisfying (3.5) is easily found to be

$$\Psi_0(\lambda) = \begin{pmatrix} e^{+iZ(\lambda)} & 0 \\ 0 & e^{-iZ(\lambda)} \end{pmatrix}, \qquad Z(\lambda) = \frac{z}{\lambda - 1} + \frac{\bar{z}}{\lambda + 1}. \tag{4.5}$$

4.2 A single two-charge soliton

Let us now dress the vacuum (4.5) to generate a one-soliton solution. We will show each step in great detail in order to demonstrate the procedure clearly. We use the dressing factor (3.9) with P given by (3.10). We can choose e to be an arbitrary constant element of $\mathbb{P}^1$, which, without loss of generality, we can parametrize as

$$e = (w, 1/w) \tag{4.6}$$

for $w \in \mathbb{C}^*$. Notice that e only enters into (3.10) in the form

$$\Psi_0(\bar{\lambda}_1) e = (\, w\, e^{+iZ(\bar{\lambda}_1)} \, \tfrac{1}{w}\, e^{-iZ(\bar{\lambda}_1)}\,). \tag{4.7}$$

It is clear now that the complex parameter w can be completely absorbed by shifting $Z(\bar{\lambda}_1) \to Z(\bar{\lambda}_1) + i \log w$. From (4.5) it is evident that such a shift amounts to some particular translation in the x and t coordinates. Since this does not substantively affect the resulting solution, we can without loss of generality go ahead and set $w = 1$ for simplicity.

The projector P can then be written as

$$P = \frac{1}{1 + e^{2i(Z(\lambda_1) - Z(\bar{\lambda}_1))}} \begin{pmatrix} 1 & e^{+2iZ(\lambda_1)} \\ e^{-2iZ(\bar{\lambda}_1)} & e^{2i(Z(\lambda_1) - Z(\bar{\lambda}_1))} \end{pmatrix}. \tag{4.8}$$

The one-soliton solution is then

$$\Psi_1(\lambda) = \left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P \right] \Psi_0(\lambda). \tag{4.9}$$

We can read off the corresponding solution in the Z_i variables from the embedding (4.1), which leads to (keeping in mind the phase discussed under (4.2))

$$\begin{aligned} Z_1 &= \frac{e^{+it}}{|\lambda_1|} \frac{\lambda_1 e^{-2iZ(\bar{\lambda}_1)} + \bar{\lambda}_1 e^{-2iZ(\lambda_1)}}{e^{-2iZ(\lambda_1)} + e^{-2iZ(\bar{\lambda}_1)}}, \\ Z_2 &= \frac{e^{-it}}{|\lambda_1|} \frac{i(\bar{\lambda}_1 - \lambda_1)}{e^{-2iZ(\lambda_1)} + e^{-2iZ(\bar{\lambda}_1)}}. \end{aligned} \tag{4.10}$$

One can check directly that this solves the string equations of motion (2.1), the Virasoro constraints (2.2), and satisfies the giant magnon boundary conditions (2.4).

It is instructive to express this solution in a more familiar form. First we parametrize

$$\lambda_1 = re^{ip/2} \tag{4.11}$$

and we introduce

$$\begin{aligned} u &= i(Z(\lambda_1) - Z(\bar{\lambda}_1)), \\ v &= Z(\lambda_1) + Z(\bar{\lambda}_1) - t, \end{aligned} \tag{4.12}$$

Plugging (4.11) into (4.12) and using (4.5), we find that u and v may be expressed as

$$\begin{aligned} u &= [x \cosh\theta - t \sinh\theta] \cos\alpha, \\ v &= [t \cosh\theta - x \sinh\theta] \sin\alpha, \end{aligned} \tag{4.13}$$

where α and θ are given by

$$\begin{aligned} \cot\alpha &= \frac{2r}{1-r^2} \sin\frac{p}{2}, \\ \tanh\theta &= \frac{2r}{1+r^2} \cos\frac{p}{2}. \end{aligned} \tag{4.14}$$

Finally, we find that the solution (4.10) may be written as

$$\begin{aligned} Z_1 &= e^{it}\left[\cos\tfrac{p}{2} + i \sin\tfrac{p}{2} \tanh u\right], \\ Z_2 &= e^{iv} \frac{\sin\frac{p}{2}}{\cosh u}. \end{aligned} \tag{4.15}$$

This form of the solution agrees precisely with the two-charge giant magnon solution in [19], where it was shown to correspond to the single-soliton solution of the complex sine-Gordon theory. As a soliton of the U(2) principal chiral model, this solution has been obtained in [26]. We also note that it reduces in the limit $r \to 1$ to the elementary giant magnon of Hofman and Maldacena [10].

If we force p to lie within the range $-2\pi < p < +2\pi$, then we see that the total momentum (2.5) is equal to $|p|$ for $-\pi < p < \pi$ and $|p| - 2\pi$ for $\pi < |p| < 2\pi$. In particular, λ_1 in the right half-plane gives a soliton and λ_1 in the left half-plane gives an anti-soliton. The charges carried by this soliton may be obtained from (2.6),

$$\begin{aligned} \Delta - J &= \frac{\sqrt{\lambda}}{\pi} \frac{1+r^2}{2r} \left|\sin\frac{p}{2}\right|, \\ J_2 &= \frac{\sqrt{\lambda}}{\pi} \frac{1-r^2}{2r} \left|\sin\frac{p}{2}\right|. \end{aligned} \tag{4.16}$$

Eliminating r between these two expressions gives the dispersion relation [20, 19]

$$\Delta - J = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2\frac{p}{2}}. \tag{4.17}$$

JHEP10(2006)012

4.3 A scattering state of two two-charge solitons

Now that we have all of the machinery set up, it is straightforward to obtain the solution corresponding to two two-charge solitons. We simply start with $\Psi_1(\lambda)$ given by (4.9) and apply the dressing method a second time, now with a pole at $\lambda = \lambda_2$. In this manner we obtain

$$\begin{aligned} Z_1 &= \frac{e^{it}}{2|\lambda_1\lambda_2|} \frac{R + |\lambda_1|^2 \lambda_{1\bar{1}} \lambda_{2\bar{2}} e^{+i(v_1 - v_2)} + |\lambda_2|^2 \lambda_{1\bar{1}} \lambda_{2\bar{2}} e^{-i(v_1 - v_2)}}{\lambda_{12}\lambda_{\bar{1}\bar{2}} \cosh(u_1 + u_2) + \lambda_{1\bar{2}}\lambda_{\bar{1}2} \cosh(u_1 - u_2) + \lambda_{1\bar{1}}\lambda_{2\bar{2}} \cos(v_1 - v_2)}, \\ Z_2 &= \frac{-i}{2|\lambda_1\lambda_2|} \frac{\lambda_{1\bar{1}} e^{iv_1} \left[\lambda_{12}\lambda_{\bar{1}\bar{2}}\bar{\lambda}_2 e^{+u_2} + \lambda_{\bar{1}\bar{2}}\lambda_{1\bar{2}}\lambda_2 e^{-u_2}\right] + (1 \leftrightarrow 2)}{\lambda_{12}\lambda_{\bar{1}\bar{2}} \cosh(u_1 + u_2) + \lambda_{1\bar{2}}\lambda_{\bar{1}2} \cosh(u_1 - u_2) + \lambda_{1\bar{1}}\lambda_{2\bar{2}} \cos(v_1 - v_2)}, \end{aligned} \tag{4.18}$$

where

$$R = \lambda_{12}\lambda_{\bar{1}\bar{2}} \left[\lambda_1\lambda_2 e^{+u_1+u_2} + \bar{\lambda}_1\bar{\lambda}_2 e^{-u_1-u_2}\right] + \lambda_{\bar{1}2}\lambda_{1\bar{2}} \left[\lambda_1\bar{\lambda}_2 e^{+u_1-u_2} + \bar{\lambda}_1\lambda_2 e^{-u_1+u_2}\right], \tag{4.19}$$

u_i and v_i are given by (4.12) with $\lambda_1 \to \lambda_i$, and we use the shorthand notation

$$\lambda_{12} = \lambda_1 - \lambda_2, \qquad \lambda_{1\bar{2}} = \lambda_1 - \bar{\lambda}_2, \qquad \text{etc.} \tag{4.20}$$

Parametrizing $\lambda_i = r_i e^{ip_i/2}$, the conserved charges of (4.18) are given by

$$\begin{aligned} \Delta - J &= \frac{\sqrt{\lambda}}{\pi} \frac{1 + r_1^2}{2r_1} \left|\sin \frac{p_1}{2}\right| + \frac{\sqrt{\lambda}}{\pi} \frac{1 + r_2^2}{2r_2} \left|\sin \frac{p_2}{2}\right|, \\ J_2 &= \frac{\sqrt{\lambda}}{\pi} \frac{1 - r_1^2}{2r_1} \left|\sin \frac{p_1}{2}\right| + \frac{\sqrt{\lambda}}{\pi} \frac{1 - r_2^2}{2r_2} \left|\sin \frac{p_2}{2}\right|. \end{aligned} \tag{4.21}$$

It is evident that (4.18) represents a scattering state composed of two solitons of the type given in (4.10) and discussed in [20, 19].

It is interesting to note that (4.18) admits a simple two (real) parameter generalization. Recall that the construction of the projector P in the dressing method requires the choice (4.6) of a vector e which we parametrized as $e = (w, 1/w)$ for some non-zero complex number $w \equiv w_1$. Previously, when we had only a single soliton, we argued that w_1 could be set to 1 without loss of generality by a suitable translation of x and t. When applying the dressing method a second time to obtain the two-soliton solution, we again have the freedom to choose a different arbitrary vector $e_2 = (w_2, 1/w_2)$, and there is no need for w_1 and w_2 to be related.

It is still true that we can absorb w_i into u_i and v_i through (4.12) for $i = 1, 2$ separately. The effect of this freedom is that (4.18) can be generalized by taking

$$u_i \to u_i + a_i, \qquad v_i \to v_i + b_i, \qquad i = 1, 2 \tag{4.22}$$

for four arbitrary real numbers a_i, b_i. Two of these parameters can be absorbed by a suitable translation of x and t, but the remaining two parameters modify the shape of the classical solution (4.18) nontrivially and therefore correspond to 'moduli' of the scattering state.

JHEP10(2006)012

The solution (4.18) can also be mapped to a two-soliton solution of the complex sine-Gordon theory by taking the angular field ϕ of CSG to be [20]

$$\cos\phi = \bar{\partial} X_i \partial X_i. \tag{4.23}$$

It would be interesting to see whether (4.18) could also be obtained by exploiting the permutativity of the Bäcklund transformation along the lines of [16]. Finally, it would be interesting to calculate the scattering phase for (4.18) (in the string theory picture), along the lines of [10].

We have demonstrated how to apply the dressing method to the problem of constructing superpositions of two-charge solitons. It is clear that this method can be used to generate n-soliton scattering solutions for any n, although the expressions are probably too cumbersome to be of great use. The generalizations of (4.21) and (4.22) to arbitrary n are obvious.

4.4 A bound state of two two-charge solitons

It is also interesting to construct a bound state of two of these two-charge solitons, along the same lines as the bound state of one-charge solitons considered in [10]. We begin by noting that in the solution (4.18), λ and $\bar{\lambda}$ are completely free parameters. The expressions given there, together with the corresponding $\bar{Z}_j$, which are obtained by taking $i \to -i$ and exchanging $\lambda_j \leftrightarrow \bar{\lambda}_j$, satisfy the equations of motion (2.1) for arbitrary complex values of λ_j and $\bar{\lambda}_j$. In order for (4.18) to be a legitimate solution of the S^3 sigma-model, however, we need to impose that the sigma-model fields X_i (2.3) are real. This can be achieved by imposing, as we usually do, that $\bar{\lambda}_j$ is the complex conjugate of λ_j. However this reality condition is also satisfied by taking λ_1 to be the complex conjugate of $\bar{\lambda}_2$ (and vice versa), a possibility that we now put to use.

For the bound state corresponding to a breather we are interested in analytically continuing p_i to complex momenta

$$p_1 = p + iq, \qquad p_2 = p - iq. \tag{4.24}$$

Using $\lambda_i = re^{+ip_i/2}$ and $\bar{\lambda}_i = re^{-ip_i/2}$ we find

$$\begin{aligned} \lambda_1 &= re^{-q/2}e^{+ip/2}, & \bar{\lambda}_1 &= re^{+q/2}e^{-ip/2}, \\ \lambda_2 &= re^{+q/2}e^{+ip/2}, & \bar{\lambda}_2 &= re^{-q/2}e^{-ip/2}, \end{aligned} \tag{4.25}$$

where we have already set $r_1 = r_2 = r$ to preserve the reality condition. We then find the 'breather' solution

$$\begin{aligned} Z_1 &= e^{it}\left[1 + \frac{\cos\frac{p}{2}\sinh\frac{q}{2}\sinh(u_1+u_2) + iN}{D}\right], \\ Z_2 &= \frac{1}{D}\left[\sin(\tfrac{p}{2} - i\tfrac{q}{2})e^{iv_2}\cosh u_1 - \sin(\tfrac{p}{2} + i\tfrac{q}{2})e^{iv_1}\cosh u_2\right], \end{aligned} \tag{4.26}$$

where

$$N = \sin\frac{p}{2}\left[2\sinh\frac{q}{2}\cosh u_1\cosh u_2 - \cosh\frac{q}{2}\sinh(u_1-u_2)\right], \tag{4.27}$$

$$D = \frac{\sinh\frac{q}{2}}{2i\sin\frac{p}{2}}\left[\cos(v_1-v_2) + \cosh(u_1+u_2)\right] + \frac{\sin\frac{p}{2}}{2i\sinh\frac{q}{2}}\left[\cos(v_1-v_2) - \cosh(u_1-u_2)\right].$$

Here u_i and v_i, which are now complex and satisfy $\bar{u}_1 = u_2$, $\bar{v}_1 = v_2$, are given in terms of (4.25) by (4.12) as usual. By taking $p \to p \pm iq$ in (4.14), one can still choose to parametrize u_i and v_i in terms of the variables α_i and θ_i, at the expense of allowing them to become complex.

This solution has charges

$$\begin{aligned} \Delta - J &= \frac{\sqrt{\lambda}}{\pi}\frac{1+r^2}{2r} 2\cosh\frac{q}{2}\left|\sin\frac{p}{2}\right|, \\ J_2 &= \frac{\sqrt{\lambda}}{\pi}\frac{1-r^2}{2r} 2\cosh\frac{q}{2}\left|\sin\frac{p}{2}\right|. \end{aligned} \tag{4.28}$$

Eliminating the parameter r gives

$$\Delta - J = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} 4\cosh^2\frac{q}{2}\sin^2\frac{p}{2}}. \tag{4.29}$$

Recall that the two-soliton scattering state is characterized by two continuous real parameters (4.22) which do not appear in the momentum p or the charges $\Delta - J$, J_2. It would be strange to find a continuous family of bound states with the same charges, and indeed we find that the reality condition on the sigma model coordinates also reduces the freedom (4.22) to a common translation $u_i \to u_i + a$, $v_i \to v_i + b$, which can be absorbed into a translation of x and t. Therefore there is a single bound state for any given p, $\Delta - J$ and J_2. This fact has a geometric interpretation: the freedom (4.22) for each soliton corresponds to the ability to choose different initial vectors (4.6) in $\mathbb{P}^1$. Thus each soliton in a multi-soliton scattering state can be 'oriented' differently inside SU(2). In order to bind together to form (4.26), their orientations inside SU(2) must align.

5. Giant magnons on $\mathbb{R} \times S^{N-1}$ from the SO(N) vector model

In this section we apply the dressing method for the SO(N) vector model to giant magnon solutions on $\mathbb{R} \times S^{N-1}$. Some of the solutions in this section may be obtained as limiting cases of the $\mathbb{R} \times S^3$ solutions obtained in the previous section, but the method described here is clearly more general since it can be applied to $\mathbb{R} \times S^{N-1}$ for $N > 4$.

5.1 The vacuum

As before (4.3), we start with the solution describing a point-like string moving at the speed of light along the equator of the sphere,

$$X_i = (\cos t, \sin t, 0). \tag{5.1}$$

For the moment we work with SO(3), describing strings on $\mathbb{R} \times S^2$. The extension to SO(N) is of course straightforward and will be employed below.

We embed (5.1) into the SO(3) principal chiral model using (3.12), and find the corresponding vacuum solution $\Psi_0(\lambda)$ to the linear system (3.3) is

$$\Psi_0(\lambda) = \begin{pmatrix} \cos 2Z(\lambda) & \sin 2Z(\lambda) & 0 \\ -\sin 2Z(\lambda) & \cos 2Z(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2}$$

where $Z(\lambda)$ is given as before by (4.5). This form of $\Psi_0(\lambda)$ has been chosen to satisfy the conditions (3.6), (3.15).

5.2 The HM giant magnon from a pair of poles on the unit circle

The simplest soliton is obtained from the dressing factor (3.18) and has two poles at conjugate points on the unit circle,

$$\lambda_1 = e^{+ip/2}, \qquad \bar{\lambda}_1 = e^{-ip/2}. \tag{5.3}$$

The projector P is given by (3.10), where we choose to parametrize the initial vector e as

$$e = (1, i\sin w, i\cos w), \tag{5.4}$$

where w is a real parameter. Up to an overall (real) scale factor, which drops out of (3.10) anyway, this is the most general choice satisfying the constraints (3.19). It turns out that w is an essentially irrelevant parameter and may be absorbed into a translation of x or t (although the analysis which leads to this conclusion is not quite as simple here as it was in the case considered under (4.7)). We can therefore set $w = 0$.

We use (5.4) and (5.2) to construct the projector P shown in (3.10) and the dressing factor $\chi(\lambda)$ given in (3.18). Then $g = \chi(0)\Psi_0(0)$ is a solution of the SO(3) principal chiral model which lives on the submanifold (3.13), so that we can use (3.12) to read off the new solution X in the S^2 sigma-model coordinates. We find

$$\begin{aligned} X_1 + iX_2 &= e^{it}\left[\cos\tfrac{p}{2} + i\sin\tfrac{p}{2}\tanh u\right], \\ X_3 &= \sin\tfrac{p}{2}\,\mathrm{sech}\, u \end{aligned} \tag{5.5}$$

with u given by

$$u = \left[x - t\cos\frac{p}{2}\right]\csc\frac{p}{2}. \tag{5.6}$$

The dispersion relation is [10]

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|. \tag{5.7}$$

This is precisely the elementary giant magnon solution of string theory on $\mathbb{R}\times S^2$ found by Hofman and Maldacena [10], and the formula (5.7) agrees with the strong coupling limit of the exact magnon dispersion relation [30, 35, 36]

$$\Delta - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}. \tag{5.8}$$

The solution (5.5) also appears in [32] as a solution of the O(3) principal chiral model, and corresponds via the map between strings on $\mathbb{R}\times S^2$ and the sine-Gordon model to a single sine-Gordon soliton. Of course this solution may also be obtained by taking the $r \to 1$ limit of (4.10), in which the single pole moves onto the unit circle and the charge J_2 goes to zero.

As mentioned above, one can check that the parameter w which we set to zero in (5.4) can be absorbed into a translation of u, which in turn can be absorbed into a translation of x or t. The generalization of this elementary soliton from $\mathbb{R}\times S^2$ to $\mathbb{R}\times S^{N-1}$ is

JHEP10(2006)012

straightforward. We parametrize (5.4) by

$$e = (1, i\sin w, i\vec{v}\cos w) \tag{5.9}$$

where $\vec{v}$ is an arbitrary $N-2$ component unit vector. As usual, the parameter w can without loss of generality be set to zero by an appropriate translation in u. The unit vector $\vec{v}$ then specifies the orientation of the HM soliton in the $N-2$ directions $(X_3, \ldots)$.

5.3 A scattering state of two HM giant magnons from two pairs of poles on the unit circle

We can further dress the solution of the previous subsection by adding a second pair of poles on the unit circle at $\lambda_2 = e^{+ip_2/2}$ and $\bar{\lambda}_2 = e^{-ip_2/2}$. This leads to the solution [29]

$$\begin{aligned} X_1 + iX_2 &= e^{it} + \frac{e^{it}(R+iI)}{\sin\frac{p_1}{2}\sin\frac{p_2}{2}(1+\sinh u_1\sinh u_2) - (1-\cos\frac{p_1}{2}\cos\frac{p_2}{2})\cosh u_1\cosh u_2}, \\ X_3 &= \frac{(\cos\frac{p_1}{2}-\cos\frac{p_2}{2})(\sin\frac{p_1}{2}\cosh u_2 - \sin\frac{p_2}{2}\cosh u_1)}{\sin\frac{p_1}{2}\sin\frac{p_2}{2}(1+\sinh u_1\sinh u_2) - (1-\cos\frac{p_1}{2}\cos\frac{p_2}{2})\cosh u_1\cosh u_2}, \end{aligned} \tag{5.10}$$

with u_i as in (5.6), and

$$\begin{aligned} R &= (\cos\tfrac{p_1}{2} - \cos\tfrac{p_2}{2})^2\cosh u_1\cosh u_2, \\ I &= (\cos\tfrac{p_1}{2} - \cos\tfrac{p_2}{2})(\sin\tfrac{p_1}{2}\sinh u_1\cosh u_2 - \sin\tfrac{p_2}{2}\cosh u_1\sinh u_2) \end{aligned} \tag{5.11}$$

This is the explicit formula for the two-soliton scattering state whose scattering phase was calculated in [10] (although the precise form was not needed there because the phase shift can easily be related to that of two solitons in the sine-Gordon model). Again (5.10) may be obtained by taking $r_1, r_2 \to 1$ in (4.18). As expected, the energy of (5.10) is

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_1}{2}\right| + \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_2}{2}\right|. \tag{5.12}$$

Continued application of the dressing method may be used to construct a scattering state with arbitrarily many solitons. Each soliton can carry a different orientation in the $N-2$ transverse directions by an appropriate choice of the initial vector (5.9), and there is always the freedom to take $u_i \to u_i + a_i$ for arbitrary real constants a_i.

5.4 A bound state of two HM solitons from four poles in the complex plane

We can also take four poles, at an arbitrary point λ_1 in the complex plane and its three images (3.17), which we parametrize as

$$\lambda_1 = e^{+i(p+iq)/2}, \qquad \bar{\lambda}_1 = e^{-i(p-iq)/2}, \qquad 1/\lambda_1 = e^{-i(p+iq)/2}, \qquad 1/\bar{\lambda}_1 = e^{+i(p-iq)/2}. \tag{5.13}$$

Following Theorem 4.2 and section 5 of [28] gives the solution [29]

$$\begin{aligned} X_1 + iX_2 &= e^{it}\frac{\sinh^2\frac{q}{2}\cosh^2(u+i\frac{p}{2}) + \sin^2\frac{p}{2}\sin^2(v+i\frac{q}{2})}{\sinh^2\frac{q}{2}\cosh^2 u + \sin^2\frac{p}{2}\sin^2 v} \\ X_3 &= \frac{\sin p\sinh^2\frac{q}{2}\cosh u\cos v - \sin^2\frac{p}{2}\sinh q\sinh u\sin v}{\sinh^2\frac{q}{2}\cosh^2 u + \sin^2\frac{p}{2}\sin^2 v} \end{aligned} \tag{5.14}$$

where

$$u = \frac{2\sin\frac{p}{2}}{\cosh q - \cos p}\left[x\cosh\frac{q}{2} - t\cos\frac{p}{2}\right],$$
$$v = \frac{2\sinh\frac{q}{2}}{\cosh q - \cos p}\left[t\cosh\frac{q}{2} - x\cos\frac{p}{2}\right]. \tag{5.15}$$

with dispersion relation

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right| 2\cosh\frac{q}{2}. \tag{5.16}$$

As evident from this dispersion relation and the fact that (5.14) is periodic in v (for fixed u), this solution represents a bound state of two HM solitons. This state was also discussed in [10] (though the full solution (5.14) was not presented).

We can also obtain the solution (5.14) by analytically continuing (5.10) as follows. We take

$$p_1 = p + iq, \qquad p_2 = p - iq \tag{5.17}$$

from which it follows that (5.15) and (5.6) are related by

$$u = \frac{1}{2}(u_1 + u_2), \qquad v = \frac{1}{2i}(u_1 - u_2). \tag{5.18}$$

Making these substitutions in (5.10) gives (5.14).

5.5 A three-charge giant magnon

The previous few subsections have demonstrated the utility of the dressing method, applied to the SO(N) vector model, for constructing giant magnons on $\mathbb{R} \times S^{N-1}$. Many of the simplest examples can be embedded inside $\mathbb{R} \times S^3$ and may therefore be obtained as limits of the U(2) principal chiral solutions we considered in the previous section.

Although the dressing method for the U(2) principal chiral model is simpler, the advantage of the SO(N) vector model is its wider applicability to $\mathbb{R} \times S^{N-1}$ for $N > 4$. We leave a thorough analysis of the general case to future work, and end here with a particularly simple example of a three-spin giant magnon on $\mathbb{R} \times S^5$. The solution is given in complex coordinates by

$$Z_1 = e^{it}\frac{\cos\alpha_1 \tanh u_1 \tanh u_2 - \cos\alpha_2}{\cos\alpha_1 - \cos\alpha_2 \tanh u_1 \tanh u_2},$$
$$Z_2 = e^{iv_1}\frac{\sqrt{\cos^2\alpha_1 - \cos^2\alpha_2}}{\cos\alpha_1 \cosh u_1 - \cos\alpha_2 \sinh u_1 \tanh u_2},$$
$$Z_3 = e^{iv_2}\frac{\sqrt{\cos^2\alpha_1 - \cos^2\alpha_2}}{\cos\alpha_1 \cosh u_2 \coth u_1 - \cos\alpha_2 \sinh u_2}, \tag{5.19}$$

where

$$u_1 = x\cos\alpha_1, \qquad v_1 = t\sin\alpha_1,$$
$$u_2 = x\cos\alpha_2, \qquad v_2 = t\sin\alpha_2. \tag{5.20}$$

This solution is valid for $\sin^2\alpha_1 < \sin^2\alpha_2$, which we can assume without loss of generality. As we have encountered before, the solution (5.19) has a four real parameter generalization given by (4.22). As usual, two of those parameters can be absorbed into shifts of x and t. In the particular case of (5.19), a third parameter can be absorbed into a rotation of Z_2 or Z_3 by a constant phase factor. The net result of this analysis is that (5.19) has a single physical modulus which adjusts the shape of the solution. This modulus may be taken to be $u_2 \to u_2 + \text{constant}$.

The solution (5.19) carries charges

$$J_2 = \frac{\sqrt{\lambda}}{\pi}\frac{\sin\alpha_1}{|\cos\alpha_1|}, \qquad J_3 = \frac{\sqrt{\lambda}}{\pi}\frac{\sin\alpha_2}{|\cos\alpha_2|} \tag{5.21}$$

and has energy

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\left(\frac{1}{|\cos\alpha_1|} + \frac{1}{|\cos\alpha_2|}\right). \tag{5.22}$$

Eliminating α_1 and α_2 gives the dispersion relation

$$\Delta - J = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}} + \sqrt{J_3^2 + \frac{\lambda}{\pi^2}}. \tag{5.23}$$

It is evident that this solution represents a scattering state consisting of two superimposed two-charge solitons (4.10), one with momentum $p = \pi$ and the other with momentum $p = -\pi$. Since the total momentum is zero, this solution is compatible with the form of the spinning string ansatz made in [37] and can be obtained directly by solving the equations of motion of the Neumann integrable system.

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